Algebraic Topology Lecture Notes (2025/2026)

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1 Introduction

1.1 ASCs

Definition *Abstract simplicial complex*

An **ASC** is a set $\mathcal{A}$ of finite sets such that for all $A \in \mathcal{A}$ and $B \subset A$, we have $B \in \mathcal{A}$.
Any ASC $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{A}$ is a **subcomplex** of $\mathcal{A}$.

Definition *Vertex set*

$$V(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$$

Definition *Dimension*

The **dimension of a set** $A \in \mathcal{A}$ is $\#A - 1$. The **dimension of an ASC** $\mathcal{A}$ is $\dim(\mathcal{A}) = \max_{A \in \mathcal{A}} \#A$.

Definition *Generated ASC*

For any finite set S, we define $\langle S \rangle$ to be the smallest ASC containing S.

Standard Möbius strip

$$M = \langle \{\{a-2, a, a+2\} : a \in \mathbb{Z}/5\mathbb{Z}\} \rangle$$

n-disk and n-sphere

$$\mathbb{D}^n = 2^{\{0,1,\ldots,n\}} \qquad \mathbb{S}^{n-1} = \mathbb{D}^n \setminus \{0, 1, \ldots, n\}$$

Closed strip, annulus and cone

$$\begin{aligned} \mathcal{CS}_n(v) &= \langle \{\{v_a, v_{a+1}, v_{a+2}\} : a \in \mathbb{Z}/n\mathbb{Z}\} \rangle \\ \mathcal{A}_n(v, w) &= \langle \{\{v_a, w_a, v_{a+1}\} : a \in \mathbb{Z}/n\mathbb{Z}\} \cup \{\{v_a, w_a, w_{a-1}\} : a \in \mathbb{Z}/n\mathbb{Z}\} \rangle \\ \mathcal{C}_n^c(v) &= \langle \{\{v_a, v_{a+1}, v_c\} : a \in \mathbb{Z}/n\mathbb{Z}\} \quad c \notin \mathbb{Z}/n\mathbb{Z} \rangle \end{aligned} \tag{1}$$

Definition *Cone on subcomplex*

For an ASC $\mathcal{A}$, subcomplex $\mathcal{B}$ and $c \notin V(\mathcal{A})$, we define the **cone** as

$$\Delta_{\mathcal{B}}^c \mathcal{A} = \mathcal{A} \cup \{B \cup \{c\} : B \in \mathcal{B}\}$$

1.2 Graphs

Definition *One-skeleton*

For an ASC $\mathcal{A}$, we define its **one-skeleton** by

$$\mathcal{A}^{(\leq 1)} = \{A \in \mathcal{A} : \#A \leq 2\}$$

Definition *Simple graph*

A **simple graph** is a 1-dimensional ASC.

Lemma *Tree lemma*

For all connected simple graphs G we have $\#V(G) - \#E(G) \leq 1$, and $\#V(G) - \#E(G) = 1$ iff G is a tree.

Definition *Connectedness*

An ASC is **connected** iff its one-skeleton is connected as a graph.

2 Surfaces

Definition *Link*

For $u \in V(\mathcal{A})$ we define the **link** or **horizon** as

$$\mathrm{Lk}(u) = \{A \in \mathcal{A} : v \notin A \text{ and } \{v\} \cup A \in \mathcal{A}\}$$

Definition *Surface*

An **surface** is an ASC $\mathcal{S}$ such that for all $u \in V(S)$, the link $\mathrm{Lk}(u)$ is either a circle graph or an interval graph.

Definition *Boundary*

The **boundary** $\partial\mathcal{S}$ of $\mathcal{S}$ is the ASC generated by all edges meeting precisely one triangle.

Theorem

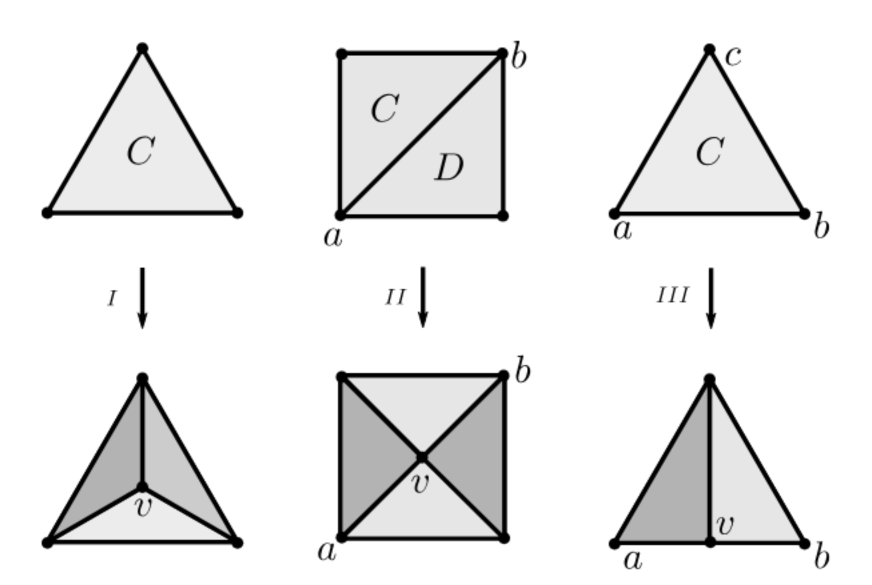
The boundary of any surface is a disjoint union of circle graphs.

2.1 Equivalence and Euler characteristic

Definition *Stellar equivalent surfaces*

Surfaces $\mathcal{S}$ and $\mathcal{S}'$ are **stellar equivalent** (denoted $\mathcal{S} \cong \mathcal{S}'$) if they can be related by finitely many **stellar moves**:

1. If C is a triangle in $\mathcal{A}$, then $\mathcal{B} = \Delta^v_{\partial\langle\{C\}\rangle}(A \setminus \{C\})$
2. If edge E meets two triangles C and D, then $\mathcal{B} = \Delta^v_{\partial\langle\{C,D\}\rangle}(A \setminus \{C, D, E\})$
3. If edge $E = \{a, b\}$ meets a unique triangle $C = \{a, b, c\}$, then $\mathcal{B} = \Delta^v_{\langle\{\{a,c\},\{b,c\}\}\rangle}(A \setminus \{C, E\})$



Definition *Euler characteristic of a surface*

The **Euler characteristic** of a surface is defined as $\chi(\mathcal{S}) = \#V(\mathcal{S}) - \#E(\mathcal{S}) + \#F(\mathcal{S})$, where $F(\mathcal{S})$ is the set of triangles in $\mathcal{S}$.

Lemma

The Euler characteristic is invariant under stellar moves.

Lemma

For any connected surface without boundary we have $\chi(S) \leq 2$

Definition *Dual graph*

The dual graph $\mathcal{D}(\mathcal{S})$ of a surface $\mathcal{S}$ is the graph $\mathcal{G}$ whose vertices are the triangles of $\mathcal{S}$. Two vertices v, w of $\mathcal{D}(\mathcal{S})$ form an edge in $\mathcal{D}(\mathcal{S})$ iff the corresponding triangles of $\mathcal{S}$ meet in an edge.

2.2 Orientation

Definition *Orientation of a triangle*

A triangle $\{a, b, c\}$ is **oriented** if we choose a cyclic order of its vertices, denoted $[a, b, c]$.

Definition *Orientation of a surface*

An **orientation of a surface** $\mathcal{S}$ is a choice of orientation for each triangle in $\mathcal{S}$ such that neighboring triangles are oriented consistently. A surface is **orientable** if there exists an orientation.

Consistent: $[0, 1, 2]$, $[1, 0, 3]$ Inconsistent: $[0, 1, 2]$, $[0, 1, 3]$

Theorem

A surface $\mathcal{S}$ is orientable if and only if $\mathcal{S}$ does not contain a Möbius strip as a subcomplex.

2.3 Connected sums

Definition *Connected sum*

The **connected sum** $\mathcal{S}\#\mathcal{S}'$ of two oriented surfaces $\mathcal{S}$ and $\mathcal{S}'$ is defined as follows.
Assuming $V(\mathcal{S}) \cap V(\mathcal{S}) \cap \{v_i, v_i'\} = \varnothing$, choose oriented triangles $T = [t_0, t_1, t_2] \in \mathcal{S}$ and $T' = [t_0', t_1', t_2'] \in \mathcal{S}'$, and define

$$\mathcal{S}\#\mathcal{S}' = ((\mathcal{S} \cup \mathcal{S}' \cup \mathcal{A}_3(v, v')) \setminus \{T, T'\}) \,/\, \sim \quad \text{where } v_i \sim t_i \text{ and } v_i' \sim t_i'$$

Lemma *Properties of connected sum*

On oriented, connected surfaces without boundary, the connected sum operation is well-defined and:

1. For connected surfaces the connected sum is a connected surface without boundary that does not depend on the chosen triangles or on the way they are glued.
2. If $\mathcal{A} \cong \mathcal{B}$ then $\mathcal{A}\#\mathcal{C} \cong \mathcal{B}\#\mathcal{C}$
3. $\mathcal{A}\#\mathcal{B} \cong \mathcal{B}\#\mathcal{A}$ (commutativity)
4. $(\mathcal{A}\#\mathcal{B})\#\mathcal{C} \cong \mathcal{A}\#(\mathcal{B}\#\mathcal{C})$ (associativity)

Theorem *Classification theorem of surfaces*

Let $\mathcal{S}$ be a connected surface with b boundary circles.
Then $\mathcal{S}$ is stellar equivalent to the connected sum of $\mathbb{S}^2$ with b holes and

- g toruses if $\mathcal{S}$ is orientable, and $\chi(\mathcal{S}) = 2 - 2g - b$
- g projective planes if $\mathcal{S}$ is not orientable, and $\chi(\mathcal{S}) = 2 - g - b$.

3 Homology

3.1 Some linear algebra

Notation

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the **finite field** with p elements. The **standard basis vectors** in $\mathbb{F}^n$ are denoted $e_1, \ldots, e_n$
Linear combinations: For any x define $\mathbb{F}x = \{\lambda x : \lambda \in \mathbb{F}\}$. It is a vector space of dimension 1 with basis x.

Definition *Direct sum of vector spaces*

$$V \oplus W = \{v + w : v \in V, w \in W\} \qquad \bigoplus_{i=1}^{n} V_i = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

Definition *Coset and quotient*

Coset: $V + w = \{v + w : v \in V\}$ **Quotient**: $V/W = \{v + W : v \in V\}$

Matrix of a linear map

We can write a linear map $L : V \to W$ as a matrix $M = (M^i_j)$ with respect to bases $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_n$. The columns of the matrix are the images of the basis vectors: $L(v_j) = \sum_i M^j_i w_i$

Note

Alternate notation for Gaussian reduction is shown on page 32 and 33 of the lecture notes.

Proposition

For any linear map $L : V \to W$, there exist bases $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_n$ of W such that

$$\text{there exists } r \textbf{ (rank) } \text{such that } L(v_i) = \begin{cases} w_i \text{ if } i \leq r \\ 0 \text{ if } i > r \end{cases}$$

3.2 First homology

Definition *q-simplex*

If an element $A \in \mathcal{A}$ of an ASC has $q + 1$ elements, it is called a q-**simplex**.
We denote the set of q-simplices in $\mathcal{A}$ by $\mathcal{A}^{(q)}$.

Definition *Euler characteristic*

$$\chi(\mathcal{A}) = \sum_{q=0}^{\infty} (-1)^q \# \mathcal{A}^{(q)}$$

Notation

$\mathbb{F}A$ denotes the vector space over $\mathbb{F}$ whose basis vectors are the elements of A.

Definition *Chain spaces*

For an ASC $\mathcal{A}$ with subcomplex $\mathcal{B} \subseteq \mathbb{A}$ define the **chain vector spaces** $C_i(\mathcal{A}, \mathcal{B})$ as follows:

1. $C_0(\mathcal{A}, \mathcal{B})$ is the vector space generated by $\mathcal{A}^{(0)}/\mathcal{B}^{(0)}$.
2. $C_1(\mathcal{A}, \mathcal{B}) = T_1/O_1$, where
   - T_1 is the vector space generated by all oriented edges in $\mathcal{A}$
   - $O_1 \subseteq T_1$ is generated by both sums of oriented edges $[a, b] + [b, a]$ and all oriented edges in $\mathcal{B}$.
3. $C_1(\mathcal{A}, \mathcal{B}) = T_1/O_1$ where
   - T_2 is the vector space generated by all oriented triangles in $\mathcal{A}$
   - $O_2 \subseteq T_2$ is generated by both sums of oriented triangles $[a, b, c] + [b, a, c]$ and all oriented triangles in $\mathcal{B}$.

Definition *Boundary maps*

$$\partial_2 : C_2(A, B) \to C_1(A, B) \qquad \partial_2([a, b, c] + O_2) = [a, b] + [b, c] + [c, a] + O_1$$
$$\partial_1 : C_1(A, B) \to C_0(A, B) \qquad \partial_1([a, b] + O_1) = \{b\} - \{a\}$$

Lemma

$$\partial_1 \circ \partial_2 = 0$$

Definition *First homology*

Let $\mathcal{A}$ be an ASC with subcomplex $\mathcal{B}$. Define:

$$Z_1(\mathcal{A},\mathcal{B}) = \ker \partial_1 \qquad B_1(\mathcal{A},\mathcal{B}) = \partial_2(C_2(\mathcal{A},\mathcal{B}))$$

The **first homology** of $\mathcal{A}$ relative to $\mathcal{B}$ with coefficients in $\mathbb{F}$ is

$$H_1(\mathcal{A},\mathcal{B};\mathbb{F}) = Z_1(\mathcal{A},\mathcal{B})/B_1(\mathcal{A},\mathcal{B})$$

Lemma

Let $\mathcal{S}$ be a surface with boundary $\partial\mathcal{S}$.

$$H_1(\mathcal{S},\partial\mathcal{S}) = H_1(\Delta^c_{\partial\mathcal{S}}\mathcal{S})$$

3.3 Simple homotopy equivalence

Definition *Simplicial map*

A **simplicial map** $f : \mathcal{A} \to \mathcal{B}$ is a map between the vertex sets of $\mathcal{A},\mathcal{B}$ such that for all $A \in \mathcal{A}$ we have $f(A) \in \mathcal{B}$. f is a **simplicial bijection** iff $f : V(\mathcal{A}) \to V(\mathcal{B})$ is invertible and $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

Inclusion map

Let $\mathcal{B}$ be a subcomplex of $\mathcal{A}$. The **inclusion map** $\iota : \mathcal{B} \to \mathcal{A}, \iota(v) = v$ is a simplicial map.

Definition *Elementary collapse and expansion*

$X \in \mathcal{A}$ is **collapsible** iff there is a unique $p \notin X$ such that $X \cup \{p\} \in \mathcal{A}$. We denote

$$\downarrow_X \mathcal{B} = \mathcal{B} \setminus \{X, X \cup \{p\}\}$$

Conversely we say $\mathcal{A} = \downarrow_X \mathcal{B}$ **expands** to $\mathcal{B}$.

Definition *Simple homotopy equivalence*

Two ASCs are **simple homotopy equivalent** ($\mathcal{A} \approx \mathcal{B}$) if they can be related by finitely many elementary collapses, elementary expansions and simplicial bijections.

Lemma

Two equivalent ASCs have the same Euler characteristic.

Lemma

Two stellar equivalent surfaces are simple homotopy equivalent.

Definition *Induced map*

For any simplicial map $f : \mathcal{A} \to \mathcal{B}$, there is a linear map $f_{*,1} : H_1(\mathcal{A}) \to H_1(\mathcal{B})$, defined by

$$f_* : C_1(\mathcal{A}) \to C_1(\mathcal{B}) \qquad f_*([x,y]) = \begin{cases} [f(x),f(y)] & \text{if } f(x) \neq f(y) \\ 0 & \text{if } f(x) = f(y) \end{cases} \text{(extended linearly)} \qquad f_{*,1}(\overline{\alpha}) = \overline{f_*(\alpha)}$$

Theorem

If $\mathcal{A} \approx \mathcal{B}$, then $H_1(\mathcal{A}) \cong H_1(\mathcal{B})$,
and $\iota_{*,1}$ (the linear map induced from the inclusion map) is an isomorphism $H_1(\mathcal{A}) \to H_1(\mathcal{B})$.

4 The fundamental group

4.1 Some group theory

Definition *Word*

Given a finite set (**alphabet**) X, a **word** in X is a finite sequence of elements in X. We write words without commas.

Definition *Free group*

Let X be set and $\widetilde{X} = \{\widetilde{x} : x \in X\}$ a set disjoint from X. The **free group** $F(X)$ on generators X is the set of equivalence classes of words in $X \cup \widetilde{X}$ with respect to the relation

$$\widetilde{x}x \sim x\widetilde{x} \sim \varnothing \qquad w \sim v \text{ if } v \text{ can be obtained by inserting } x\widetilde{x} \text{ or } \widetilde{x}x \text{ in } w$$

Definition *Group presentation*

Given a finite set X and a set R of words in $X \cup \widetilde{X}$, define $\langle X \mid R \rangle$ to be the set of equivalence classes of words in $X \cup \widetilde{X}$ where $w \sim v$ if v can be obtained by inserting words from $R \cup \{x\widetilde{x} \mid x \in X\} \cup \{\widetilde{x}x \mid x \in X\}$

Theorem *Tietze's theorem*

Presentations $\langle X \mid R \rangle$ and $\langle X \mid \widetilde{R} \rangle$ are isomorphic if and only if they are related by finitely many **Tietze moves**:

1. Include a new relation that is a consequence of the relations.
2. Remove a relation that is a consequence of the other relations.
3. Include a new generator y together with the relation $y^{-1}w$, for some word in the existing generators.
4. If there is a relation of the form $y^{-1}w$ for some word w in the other generators, remove y from the list of generators and replace each occurrence of y in the other relations by w.

Proposition

All finite groups have a presentation. If you allow X, R to be infinite, then all groups have a presentation.

Examples of group presentations

$$\langle g_1, g_2, \ldots, g_n \mid \varnothing \rangle = F_n \qquad \textbf{(free group)}$$
$$\langle g_1, g_2, \ldots, g_n \mid g_i g_j g_i^{-1} g_j^{-1} : i, j = 1, \ldots, n \rangle \cong \mathbb{Z}^n \qquad (\textbf{Abelianization} \text{ of the free group})$$
$$\langle g \mid g^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$$

4.2 The fundamental group

Definition *Path*

A **path** in an ASC $\mathcal{A}$ is a sequence of vertices $p_1, \ldots, p_n$ such that $\{p_i, p_{i+1} \in \mathcal{A}\}$.

Definition *Equivalence of paths*

Two paths α, β in $\mathcal{A}$ are equivalent if $\mathcal{A}$ can be obtained by applying a finite number of the following replacements:

1. $(\ldots, v, \ldots) \leftrightarrow (\ldots, v, v, \ldots)$
2. $(\ldots, v, \ldots) \leftrightarrow (\ldots, v, u, v, \ldots)$, provided $\{v, u\} \in \mathcal{A}^{(1)}$
3. $(\ldots, v, w, \ldots) \leftrightarrow (\ldots, v, u, w, \ldots)$, provided $\{v, u, w\} \in \mathcal{A}^{(2)}$

Definition *Fundamental group*

The **fundamental group** $\pi_1(\mathcal{A}, b)$ of an ASC $\mathcal{A}$ with **base point** b is the group of equivalence classes of paths starting and ending at b. We define multiplication by concatenation, and the unit element is the constant path.

Theorem

If $\mathcal{A}^{(\leq 2)} = \mathcal{B}^{(\leq 2)}$, then the fundamental groups of $\mathcal{A}, \mathcal{B}$ will be isomorphic.

Theorem *Presentation of the fundamental group*

Let $\mathcal{A}$ be a connected ASC and choose a maximal tree T. For $b \in V(\mathcal{A})$, $\pi_1(\mathcal{A}, b) \cong \langle X \mid R \rangle$, where

$$X = \{g_{[a,b]} : [a,b] \in \mathcal{A}\} \qquad R = \{g_{[u,v]}g_{[v,w]}g_{[w,u]} : \{u,v,w\} \in \mathcal{A}\} \cup \{g_{[u,v]} : \{u,v\} \in T\}$$

Definition

For a connected ASC $\mathcal{A}$, $b \in V(\mathcal{A})$, and spanning tree T,
define γv to be the unique path in T from v to b not repeating vertices.

Proposition

Consider $\langle X \mid R \rangle$ from the presentation theorem. The following map is an isomorphism:

$$h : \langle X \mid R \rangle \to \pi_1(\mathcal{A}, b) \qquad h(g_{[u,v]}) = \overline{\gamma y \gamma^{-1} v}$$

4.3 Invariance

Proposition *Induced homomorphism*

Given a simplicial map $f : \mathcal{A} \to \mathcal{B}$, there is an **induced homomorphism**

$$f_* : \pi_1(\mathcal{A}, b) \to \pi_1(\mathcal{B}, f(b)) \qquad f_*(\overline{(v_0, v_1, \ldots, v_n)}) = \overline{(f(v_0), f(v_1), \ldots, f(v_n))}$$

Theorem

If $\mathcal{D} \approx \mathcal{C}$ (and b is sent to $\tilde{b}$), then $\pi_1(\mathcal{C}, b) \cong \pi_1(\mathcal{D}, \tilde{b})$

Lemma *Changing the base point*

$\pi_1(\mathcal{A}, b) \cong \pi_1(\mathcal{A}, b')$ whenever there is a path β connecting b to b'. The isomorphism is $f(\alpha) = \beta^{-1}\alpha\beta$.

Definition *Abelianization*

For any group G, the **commutator subgroup** $[G, G]$ is the subgroup generated by all **commutators**:

$$[[g, h]] = ghg^{-1}h^{-1} \qquad g, h \in G$$

$G/[G, G]$ is the **Abelianization** of G.

Theorem

$$H_1(\mathcal{A}; \mathbb{Z}) \cong \pi_1(\mathcal{A}, b) \,/\, [\pi_1(\mathcal{A}, b), \pi_1(\mathcal{A}, b)]$$

5 Covering spaces

5.1 Covering spaces

Definition *Star*

The **star** of $v \in V(\mathcal{A})$ is

$$\mathrm{Star}_{\mathcal{A}}(v) = \{A \in \mathcal{A} : \{v\} \cup A \in \mathcal{A}\}$$

Definition *Covering space*

A **covering** (space) of an ASC $\mathcal{B}$ is a simplicial map $p : \mathcal{Y} \to \mathcal{B}$ for some ASC $\mathcal{Y}$ satisfying the following properties for all $v \in V(\mathcal{B})$:

1. For all $w \in p^{-1}(\{v\})$, the restriction $p|_{\mathrm{Star}_{\mathcal{Y}}(w)} : \mathrm{Star}_{\mathcal{Y}}(w) \to \mathrm{Star}_{\mathcal{B}}(v)$ is a simplicial isomorphism.
2. For all $w, w' \in p^{-1}(\{v\})$, $\mathrm{Star}_{\mathcal{Y}}(w) \cap \mathrm{Star}_{\mathcal{Y}}(w') = \varnothing$
3. $p^{-1}(\mathrm{Star}_{\mathcal{B}}(v)) = \bigcup_{w \in p^{-1}(\{v\})} \mathrm{Star}_{\mathcal{Y}}(w)$

Lemma *Path lifting*

Let $p : \mathcal{Y} \to \mathcal{B}$ be a covering.
For any path $\beta = (\beta_0, \ldots, \beta_n)$ in $\mathcal{B}$ and $\tilde{\beta}_0 \in p^{-1}(\{\beta_0\})$ there is a unique path $\tilde{\beta}$ starting at $\tilde{\beta}_0$ in $\mathcal{Y}$.

Proposition

Let $p : \mathcal{Y} \to \mathcal{B}$ be a covering. If $\mathcal{B}$ is connected, then $\#p^{-1}(\{v\})$ does not depend on $v \in V(\mathcal{B})$.
The cardinality of $p^{-1}(\{v\})$ is called the **number of sheets**.

Definition *Monodromy action*

$\pi_1(\mathcal{B}, b)$ acts from the right on $p^{-1}(\{b\})$ as follows:

$$\tilde{b} = x \cdot [\alpha] = \text{ endpoint of the lift } \tilde{\alpha} \text{ of } \alpha \text{ starting at } x$$

Definition *Standard covering*

Imagine a connected ASC $\mathcal{B}$ with base point b and set $G = \pi_1(\mathcal{B}, b)$. For any subgroup $H < G$ define an ASC $\mathcal{B}_H$ by setting $V(\mathcal{B}_H) = V(\mathcal{B}) \times G/H$ and $p : V(\mathcal{B}_H) \to V(\mathcal{B})$ by $p(b, x) = b$:

$$\mathcal{B}_H = \{A \subseteq V(\mathcal{B}) \times G/H : xg_{[b,b']} = x',\ \forall (b, x), (b', x') \in A \text{ and } p(A) \in \mathcal{B}\}$$

5.2 Galois correspondence

Definition *Covering morphism*

Let $p : \mathcal{Y} \to \mathcal{B}$ and $q : \mathcal{Z} \to \mathcal{B}$ be coverings.
We say that $f : \mathcal{Y} \to \mathcal{Z}$ is a $\mathcal{B}$-**morphism** or **covering morphism** if $q \circ f = p$.
If f is a simplicial isomorphism and both f and its inverse are $\mathcal{B}$-morphisms we say f is a $\mathcal{B}$-**isomorphism**.
We denote a $\mathcal{B}$-isomorphism by $\mathcal{Y} \overset{\mathcal{B}}{\cong} \mathcal{Z}$ and the set of $\mathcal{B}$-isomorphisms from $\mathcal{Y}$ to itself by $\mathrm{Aut}(\mathcal{Y})$.

Theorem *Galois correspondence theorem*

Let $\mathcal{B}$ be a connected ASC and $b \in V(\mathcal{B})$. Let CS be the set of conjugacy classes of subgroups of $G = \pi_1(\mathcal{B}, b)$, and CC the set of $\mathcal{B}$-isomorphism classes of connected coverings of $\mathcal{B}$. There is a well-defined bijection:

$$J : CC \to CS \qquad H = p_*(\pi_1(\mathcal{Y}, \tilde{b})) = J(\mathcal{Y} \xrightarrow{p} \mathcal{B}) \qquad \tilde{b} \in p^{-1}(\{b\})$$

Moreover, the number of sheets of $J(\mathcal{Y})$ equals $[G : H]$, and if H is normal then $G/H \cong \mathrm{Aut}(\mathcal{Y})$.

Theorem *Lifting criterion*

Let $\mathcal{Z}$ be a connected ASC, $p : \mathcal{Y} \to \mathcal{B}$ a covering, $q : \mathcal{Z} \to \mathcal{B}$ an simplicial map, $\tilde{b} \in p^{-1}(\{b\})$ and $w \in q^{-1}$. The following are equivalent:

1. There exists a unique simplicial map $\tilde{q} : \mathcal{Z} \to \mathcal{Y}$ such that $p \circ \tilde{q} = q$ and $\tilde{q}(w) = \tilde{b}$.
2. $q_* \pi_1(\mathcal{Z}, w)$ is a subgroup of $p_* \pi_1(\mathcal{Y}, \tilde{b})$.

Lemma *Cover morphism uniqueness*

If $p : \mathcal{Y} \to \mathcal{B}$ is a connected covering and $\phi : \mathcal{Y} \to \mathcal{Y}$ is a covering morphism such that $\phi(y) = y$ for some $y \in V(\mathcal{Y})$ then ϕ must be the identity map.

6 Higher homology

6.1 Chain complexes

Definition *Chain complex*

A **chain complex** is a sequence of vector spaces and linear maps

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_{q-1} \circ \partial_q = 0$ for all q.

Definition *Orientation of a q-simplex*

An **orientation** of a q-simplex S is a choice of equivalence class where

$$\{a_0, \ldots, a_n\} \sim \{a_{\partial(0)} \ldots a_{\partial(n)}\} \qquad \text{iff } \partial \text{ is an even permutation}$$

Definition *Oriented q-chain*

Let T_q be the vector space generated by oriented q-simplices, σ a permutation, and O_q the vector subspace generated by $[a_0, a_1, \ldots, a_n] + \operatorname{sign}(\sigma)[a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(n)}]$ for all $\{a_0, a_1, \ldots, a_n\} \in \mathcal{A}$. Then we define $C_q(\mathcal{A}; \mathbb{F}) = T_q / O_q$.

Definition *Boundary map*

$$\partial_q : C_q(\mathcal{A}) \to C_{q-1}(\mathcal{A}) \qquad \partial_q([a_0, \ldots, a_q]) = \sum_{i=0}^{q} (-1)^i [a_0, \ldots, \hat{a}_i, \ldots, a_q]$$

The hat denotes that we remove a_i.

Lemma

$\partial_{q-1} \circ \partial_q = 0$, and thus C_q and ∂_q form a chain complex.

6.2 q-th homology

Definition *q-th homology*

The q-**th homology** of a complex C is

$$H_q(C) = \ker \partial_q / \operatorname{im} \partial_{q+1}$$

Lemma

If $\mathcal{A} \approx \mathcal{B}$, then $H_q(\mathcal{A}) \cong H_q(\mathcal{B})$

Note

$H_0(\mathcal{A})$ measures the path components of $\mathcal{A}$.

Theorem

For any ASC $\mathcal{A}$ we have

$$\sum_{q=0}^{\infty} \dim H_q(\mathcal{A};\mathbb{Q}) = \chi(\mathcal{A})$$

Theorem

$\mathcal{A}$ is orientable if and only if $H_2(\mathcal{A},\partial\mathcal{A};\mathbb{Q}) \neq 0$.

Definition *Induced map*

Let $f:\mathcal{A}\to\mathcal{B}$ be a simplicial map. We have the linear map $f_*: C_q(\mathcal{A})\to C_q(\mathcal{B})$ defined by:

$$f_*([v_0,\ldots,v_q]) = \begin{cases} \overline{[f(v_0),\ldots,f(v_q)]} & \text{if } \dim f(\{v_0,\ldots,v_q\}) = q \\ 0 & \text{if } \dim f(\{v_0,\ldots,v_q\}) \neq q \end{cases} \quad \text{(extended linearly)}$$

6.3 Mayer-Vietoris exact sequence

Definition *Exact sequence*

A sequence of maps f_n is **exact** if $\ker f_q = \operatorname{im} f_{q+1}$ for all q.

Theorem *Mayer-Vietoris exact sequence*

For any ASC $\Sigma = \Gamma\cup\Phi$ with $\dim\Sigma = n$, we define the following maps for any q:

$$H_q(\Gamma\cap\Phi) \xrightarrow{j_q} H_q(\Gamma)\oplus H_q(\Phi) \xrightarrow{k_q} H_q(\Gamma\cup\Phi) \xrightarrow{\ell_q} H_{q-1}(\Gamma\cap\Phi)$$

Let ι be the inclusion map, and define j_i and k_i by:

$$j_q = (\iota_\Gamma)_* \oplus (\iota_\Phi)_* \qquad k_q(\overline{\alpha}\oplus\overline{\beta}) = \overline{\alpha-\beta}$$

and define ℓ_i by:

$$x = \overline{\gamma}+\overline{\varphi} \qquad \gamma\in C_i(\Gamma) \qquad \phi\in C_i(\Phi) \qquad \ell_i(x) = \overline{\partial_i(\gamma)} \in H_{i-1}(\Gamma\cap\Phi)$$

The following sequence is exact:

$$\begin{aligned} 0 \to H_n(\Gamma\cap\Phi) \xrightarrow{j_n} H_n(\Gamma)\oplus H_n(\Phi) \xrightarrow{k_n} H_n(\Gamma\cup\Phi) \xrightarrow{\ell_n} \\ H_{n-1}(\Gamma\cap\Phi) \xrightarrow{j_{n-1}} H_{n-1}(\Gamma)\oplus H_{n-1}(\Phi) \xrightarrow{k_{n-1}} H_{n-1}(\Gamma\cup\Phi) \xrightarrow{\ell_{n-1}} \\ \cdots \\ \to H_q(\Gamma\cap\Phi) \xrightarrow{j_q} H_q(\Gamma)\oplus H_q(\Phi) \xrightarrow{k_q} H_q(\Gamma\cup\Phi) \xrightarrow{\ell_q} \\ \cdots \\ \to H_0(\Gamma\cap\Phi) \xrightarrow{j_0} H_0(\Gamma)\oplus H_0(\Phi) \xrightarrow{k_0} H_0(\Gamma\cup\Phi) \xrightarrow{\ell_0} 0 \end{aligned}$$

7 Topology

7.1 Topological spaces

Definition *Topological space*

A **topology** $\mathcal{T}$ on a set X is a subset $\mathcal{T}\subseteq 2^X$ satisfying

1. $\mathcal{T}$ contains $\varnothing$ and X.
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions

We call sets in $\mathcal{T}$ **open** and sets in $\mathcal{T}^c$ **closed**. The pair $(X,\mathcal{T})$ is called a **topological space**.

Definition *Continuity*

A function $f : X \to Y$ is **continuous** iff for all open sets $O \subseteq Y$, $f^{-1}(O)$ is open in X.
A bijection whose inverse is also continuous is called a **homeomorphism**.

Definition *Subspace topology*

If $B \subseteq X$, then a topology $\mathcal{T}$ on X gives a topology on B, defined by $\mathcal{T}_B = \{T \cap B \mid T \in \mathcal{T}\}$

7.2 Polyhedra

Definition *Standard simplex*

For any finite set S define the vector space $\mathbb{R}S$ spanned by vectors $\{e_s : s \in S\}$
and the **standard simplex** of dimension $\#S$:

$$\Delta_S = \left\{ \sum_{s \in S} a_s s : \sum_{s \in S} a_s = 1, a_s \in [0, 1] \right\}$$

Definition *Polyhedron*

For an ASC $\mathcal{A}$ we define the **polyhedron** of $\mathcal{A}$ to be

$$|\mathcal{A}| = \bigcup_{A \in \mathcal{A}} \Delta_A \subseteq \mathbb{R}V(\mathcal{A})$$

$|\mathcal{A}|$ is a topological space with respect to the subspace topology .

Definition

Given a simplicial map $g : \mathcal{A} \to \mathcal{B}$ we define $|g| : |\mathcal{A}| \to |\mathcal{B}|$ by

$$|g| \left(\sum_{a \in A} \lambda_a a \right) = \sum_{a \in A} \lambda_a g(a)$$

7.3 Homotopy

Definition *Homotopy*

Given continuous maps $f, g : X \to Y$ we say f and g are **homotopic**, denoted $f \simeq g$,
if there is a map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$

Straight line homotopy

If Y is a vector space, we have the following homotopy:

$$H(x, t) = f(x)(1 - t) + tg(x)$$

Definition *Homotopy equivalence*

Topological spaces X, Y are **homotopy equivalent** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are simple homotopy equivalent, then $|\mathcal{A}|$ and $|\mathcal{B}|$ are homotopy equivalent.

7.4 Simplicial approximation

Definition *Barycentric subdivision*

Given an ASC $\mathcal{A}$ define

$$\mathcal{A}' = \{(A_0, A_1, \ldots,) : A_i \in \mathcal{A}, A_i \subsetneq A_{i+1}\}$$

Note that we have $V(\mathcal{A}') = \mathcal{A}$. We denote n-fold subdivision by $\mathcal{A}'^n$.

Definition *Carrier*

For all $x \in |\mathcal{A}|$ there is a unique $A \in \mathcal{A}$ with maximal dimension such that $x \in \Delta_A$.
We call $\Delta_A = \mathrm{carr}(x)$ the **carrier** of x.

Definition *Simplicial approximation*

$g : \mathcal{A} \to \mathcal{B}$ is a simplicial approximation of $f : \mathcal{A} \to \mathcal{B}$ iff for all $x \in |\mathcal{A}|$, $|g|(x) \in \mathrm{carr}(f(x))$

Lemma

f is homotopic to $|g|$ if g approximates f.

Theorem *Simplicial approximation theorem*

For any continuous map $f : |\mathcal{A}| \to |\mathcal{B}|$ there exists a simplicial approximation $g : \mathcal{A}'^n \to \mathcal{B}$ of the map $f \circ h : |\mathcal{A}'^n| \to |\mathcal{B}|$ where h is the standard homeomorphism $|\mathcal{A}'^n| \to |\mathcal{A}|$.

Lemma *Barycentric coordinates*

Let $v_0, \ldots, v_n$ be vectors in V such that removing any v_i gives a basis of V.
For every $x \in S = \left\{ y \in v : y = \sum_{i=0}^{n} \lambda_i v_i \right\}$ there exist unique $\lambda_1, \ldots, \lambda_n$ such that $x = \sum_{i=0}^{n} \lambda_i v_i$, satisfying $\sum_{i=0}^{n} \lambda_i = 1$.
The numbers λ_i are called the **barycentric coordinates** of x w.r.t. $v_0, \ldots, v_n$.
The point with $\lambda_i = \frac{1}{n+1}$ called the **barycentre** of the triangle $[v_0, \ldots, v_n] = \{x \in S : \lambda_i \in [0, 1]\}$.

Theorem *Brouwer fixed point theorem*

For any continuous map $f : |\mathcal{D}^n| \to |\mathcal{D}^n|$ there exists $x \in |\mathcal{D}^n|$ with $f(x) = x$.

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